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LETTER TO THE EDITOR

On the critical exponent ν for c -animals

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Abstract. We study the exponent ν_c characterizing the mean-square radius of gyration of lattice bond c -animals with cyclomatic index c . We prove that $\nu_c = \nu_0$, where ν_0 is the exponent for lattice bond trees.

Lattice animals have been studied as models of branched polymers with excluded volume constraints. Of particular interest has been the influence of cycles on the properties of lattice animals. In particular, previous work (Lubensky and Isaacson 1979, Family 1980, Lam 1987) has indicated that the critical exponent ν is independent of cycle fugacity or cyclomatic index c . In this letter, we prove that for lattice bond c -animals, $\nu_c = \nu_0$, where ν_0 is the exponent for lattice bond trees.

A lattice bond c -animal is a connected subgraph on a regular d -dimensional lattice with cyclomatic index c . The cyclomatic index is the number of independent cycles, or the maximum number of edges which can be removed without disconnecting the animal. When $c=0$, the animal is called a tree. Denoting by $a_n(c)$, the number of bond c -animals with n vertices, Whittington *et al* (1983) have shown that, for all fixed c ,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \ln a_n(c) = \lim_{n \rightarrow \infty} \frac{1}{n} \ln t_n = \ln \lambda_0 \tag{1}$$

where t_n is the number of bond trees with n vertices. Moreover, the expected asymptotic form for $a_n(x)$ is

$$a_n(c) \sim n^{-\theta_c} \lambda_0^n \tag{2}$$

where $\theta_c = \theta_0 - c$ (Soteros and Whittington 1988).

The quantity we focus on is the mean-square radius of gyration of lattice bond c -animals defined by

$$\langle R_n^2 \rangle = \sum_{i, r_i} r_i^2 a_n(c, r_i) \left(n \sum_{r_i} a_n(c, r_i) \right)^{-1} \tag{3}$$

where $a_n(c, r_i)$ is the number of n -vertex lattice c -animals with the i th vertex at a distance r_i from the centre-of-mass and for fixed i , $\sum_{r_i} a_n(c, r_i) = a_n(c)$, the total number of c -animals with n vertices. In the limit $n \rightarrow \infty$, one expects that

$$\langle R_n^2 \rangle \sim n^{2\nu_c} \tag{4}$$

We show that for any fixed c

$$\nu_c = \nu_0 \tag{5}$$

if either of them exists. The result is obtained by using a pattern theorem for lattice animals due to Madras (1988). We first briefly describe the theorem and obtain a corollary.

Let L be the simple hypercubic lattice in R^d . We denote by \mathcal{T}_n the set of bond trees with n vertices. $P = (P_1, P_2)$ is defined as a *proper pattern* for \mathcal{T}_n if P_1 and P_2 are disjoint finite subsets of L (vertices and/or bonds) such that for all sufficiently large n , there exists a tree T in \mathcal{T}_n which contains all of P_1 and none of P_2 . Madras (1988) has proved the following theorem:

Theorem. Let $t_n(\varepsilon, \leq)$ be the number of n -vertex bond trees in which a proper pattern P occurs at most εn times. Then

$$\limsup_{n \rightarrow \infty} t_n(\varepsilon_0, \leq)^{1/n} < \lambda_0 \quad (6)$$

for some $\varepsilon_0 > 0$.

A corollary from the theorem is as follows.

Corollary. Let P be a proper pattern. There exists a positive number ε_0 such that if $t_n(\varepsilon_0, >)$ is the number of n -vertex bond trees in which the pattern P occurs more than $\varepsilon_0 n$ times, then

$$\lim_{n \rightarrow \infty} \frac{t_n(\varepsilon_0, >)}{t_n} = 1. \quad (7)$$

Proof. Let P be a proper pattern for bond trees. From the theorem, there exist $\varepsilon_0 > 0$, $\delta > 0$ and an integer $N(\delta)$ such that for any $n > N(\delta)$, we have

$$0 < t_n(\varepsilon_0, \leq) < (\lambda_0 - \delta)^n. \quad (8)$$

Since

$$\frac{t_n(\varepsilon_0, >)}{t_n} = 1 - \frac{t_n(\varepsilon_0, \leq)}{t_n}. \quad (9)$$

By using (8), we obtain that

$$1 - \frac{(\lambda_0 - \delta)^n}{t_n} < \frac{t_n(\varepsilon_0, >)}{t_n} < 1. \quad (10)$$

Letting $n \rightarrow \infty$ yields

$$1 \leq \liminf_{n \rightarrow \infty} \frac{t_n(\varepsilon_0, >)}{t_n} \leq \limsup_{n \rightarrow \infty} \frac{t_n(\varepsilon_0, >)}{t_n} \leq 1 \quad (11)$$

which gives (7).

By using the pattern theorem and its corollary, we prove (5) as follows.

It has been shown by Whittington *et al* (1983) that by deleting an edge from a cycle of a c -animal, one converts the c -animals into a $(c-1)$ -animal and the resulting $(c-1)$ -animal can have at most $2dn$ c -animals as its precursors. In this procedure the position of all n vertices of the c -animal on the lattice is unchanged. Therefore, denoting

by $t_n(r_i)$, the number of n -vertex trees in which the i th vertex is a distance r_i from the centre-of-mass, we obtain

$$a_n(c, r_i) \leq (2dn)a_n(c-1, r_i) \tag{12}$$

or

$$a_n(c, r_i) \leq (2dn)^c t_n(r_i) \tag{13}$$

and therefore, for fixed i ,

$$\sum_{r_i} a_n(c, r_i) \leq (2dn)^c \sum_{r_i} r_n(r_i). \tag{14}$$

Let P_1 be \sqsupset and P_2 the complement of P_1 in \square , the elementary square. From Whittington and Soteris (1990), $P=(P_1, P_2)$ is a proper pattern for bond trees. At such a pattern, by adding the complement edge to P_1 , we convert it into a cycle, \square . Therefore, if an n -vertex bond tree contains more than εn patterns P for $\varepsilon > 0$, there are $\binom{\varepsilon n}{c}$ ways to choose c of them with $\varepsilon n \geq c$. By converting each of the c chosen patterns into a cycle, we convert the tree into a c -animal. In this procedure, the position of the i th vertex on the lattice is unchanged. Denoting by $t_n(\varepsilon, >, r_i)$ the number of n -vertex trees in which the pattern P occurs more than εn times for $\varepsilon > 0$, we obtain

$$a_n(c, r_i) \geq \binom{\varepsilon n}{c} t_n(\varepsilon, >, r_i) \tag{15}$$

and for fixed i

$$\sum_{r_i} a_n(c, r_i) \geq \binom{\varepsilon n}{c} \sum_{r_i} t_n(\varepsilon, >, r_i). \tag{16}$$

From (3), (13)-(16), we have

$$\frac{\binom{\varepsilon n}{c} \sum_{i, r_i} r_i^2 t_n(\varepsilon, >, r_i)}{(2dn)^c n \sum_{r_i} t_n(r_i)} \leq \frac{\sum_{i, r_i} r_i^2 a_n(c, r_i)}{n \sum_{r_i} a_n(c, r_i)} \leq \frac{(2dn)^c \sum_{i, r_i} r_i^2 t_n(r_i)}{\binom{\varepsilon n}{c} n \sum_{r_i} t_n(\varepsilon, >, r_i)} \tag{17}$$

Since

$$\begin{aligned} \sum_{i, r_i} r_i^2 t_n(\varepsilon, >, r_i) &= \sum_{i, r_i} r_i^2 t_n(r_i) - \sum_{i, r_i} r_i^2 t_n(\varepsilon, \leq, r_i) \\ &\geq \sum_{i, r_i} r_i^2 t_n - n^3 t_n(\varepsilon, \leq) \end{aligned} \tag{18}$$

and

$$t_n(\varepsilon, >) = \sum_{r_i} t_n(\varepsilon, >, r_i) = \sum_{r_i} [t_n(r_i) - t_n(\varepsilon, \leq, r_i)] = t_n - t_n(\varepsilon, \leq) \tag{19}$$

then from the theorem and the corollary, there exists $\varepsilon_0 > 0$ such that

$$0 \leq \lim_{n \rightarrow \infty} \frac{\sum_{i, r_i} r_i^2 t_n(\varepsilon_0, \leq, r_i)}{t_n} \leq \lim_{n \rightarrow \infty} \frac{n^3 t_n(\varepsilon_0, \leq)}{t_n} = 0 \tag{20}$$

and

$$\lim_{n \rightarrow \infty} \frac{\sum_{r_i} t_n(\varepsilon_0, >, r_i)}{\sum_{r_i} t_n(r_i)} = \lim_{n \rightarrow \infty} \frac{t_n(\varepsilon_0, >)}{t_n} = 1. \tag{21}$$

Hence, for any $\varepsilon \leq \varepsilon_0$ and n sufficiently large, we have

$$\frac{\binom{\varepsilon n}{c} \sum_{i,r_i} r_i^2 t_n(r_i)}{(2dn)^c n \sum_{r_i} t_n(r_i)} \leq \frac{\sum_{i,r_i} r_i^2 a_n(c, r_i)}{n \sum_{r_i} a_n(c, r_i)} \leq \frac{(2dn)^c \sum_{i,r_i} r_i^2 t_n(r_i)}{\binom{\varepsilon n}{c} n \sum_{r_i} t_n(r_i)} \quad (22)$$

which implies that $\nu_c = \nu_0$.

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